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From the Descent Algebra to the Peak Algebra on the Symmetric Group

Pauline Hubert and Nadia Lafrenière
Laboratoire de combinatoire et d'informatique mathématique, Université du Québec à Montréal

Given a permutation σ in the symmetric group of order n, the descents are the positions $\{i \mid \sigma(i) > \sigma(i+1)\}$. This leads to an algebra whose elements are sums of permutations sharing the same descent set. This algebra has been widely studied for its connections with Coxeter groups and card shuffling. In a similar way, the peaks correspond to the set $\{i \mid \sigma(i-1) < \sigma(i) > \sigma(i+1)\}$. This allows us to define a subalgebra of the descent algebra, that is as well a left ideal of it. We extend results on the descent algebra to their equivalent in terms of peaks.

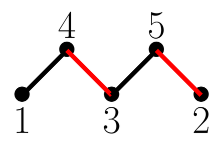
The Descent Algebra

Given a permutation $\sigma \in \mathfrak{S}_n$, the **descents** of σ is the set of positions

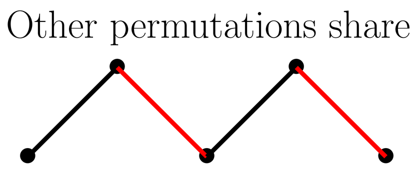
$$\mathrm{Des}(\sigma) = \{i \in \{1, \ldots, n-1\} \mid \sigma(i) > \sigma(i+1)\}.$$

To any descent set, one can associate a nonempty set of permutations corresponding to it.

Example. The permutation 14352 has descents in positions 2 and 4:



Other permutations share the same descent set as well:



13245, 14253, 15243, 15342, 23154, 24153, 24351,
25143, 25341, 34152, 34251, 35142, 35241, 45132, 45231

Example. Permutations classified by their descents in $\mathfrak{S}_4$

$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
1234	2134	1324	1243	3214	2143	1432	4321
	3124	1423	1342	4213	3142	2431	
	4132	2314	2341	3241	4312	3421	
		2413		4132			
		3412		4231			

Definition 1. *For each $S \subseteq \{1, 2, \ldots, n-1\}$, we define $\mathcal{D}_n(S) = \sum_{\mathrm{Des}(\sigma)=S} \sigma$.*

*The **descent algebra** is the span of the permutations in descent classes:*

$$\mathcal{D}_n = \mathrm{span}(\mathcal{D}_n(S) \mid S \subseteq \{1, 2, \ldots, n-1\}).$$

Example. The element $\mathcal{D}_4(\{2\})$ belongs to $\mathcal{D}_4$: $1324 + 1423 + 2314 + 2413 + 3412 \in \mathcal{D}_4$.

Multiplying two elements of $\mathcal{D}_4$ returns an element of $\mathcal{D}_4$:

$$\mathcal{D}_4(\{2\}) \cdot \mathcal{D}_4(\{1,2\}) = \mathcal{D}_4(\{1\}) + \mathcal{D}_4(\{1,2\}) + \mathcal{D}_4(\{1,2,3\}) + \mathcal{D}_4(\{2\}) + \mathcal{D}_4(\{2,3\}).$$

Since it is stable under multiplication and addition, this forms an **algebra**.

Applications of the Descent Algebra

Magic tricks

Riffle shuffle bounds the number of *rising sequences*:

$$r(\sigma) = \#\,\mathrm{Des}(\sigma^{-1}) + 1.$$

After k shuffles, there are at most 2^k rising sequences. Thus, after one shuffle:

3	1	5	2	4

Possible - 2 rising sequences

3	5	2	4	1

Not possible - 3 rising sequences

Computing $\#\{\sigma \in \mathfrak{S}_n \mid \mathrm{Des}(\sigma) = S\}$

Let $\alpha_n(S)$ be the number of permutations of $\mathfrak{S}_n$ which descent set is contained in S and let $\beta_n(S)$ be the number of permutations of $\mathfrak{S}_n$ which have exactly S as descent set. Then, by inclusion-exclusion:

$$\beta_n(S) = \sum_{T \subseteq S} (-1)^{\#(S-T)} \alpha_n(T)$$

Let S' be the ordered list of the elements of S, $S' = (1 \le s_1 < s_2 < \cdots < s_k \le n-1)$, and let $s_0 = 0$ and $s_{k+1} = n$. To build a permutation whose descent set is contained in S, for each couple of descents (s_i, s_{i+1}) $(0 \le i \le k)$, we choose $s_{i+1} - s_i$ elements and put them in increasing order. This implies that

$$\alpha_n(S) = \binom{n}{s_1, s_2 - s_1, \cdots, n - s_k}.$$

After calculations, we get the following formula for $\beta_n(S)$:

$$\beta_n(S) = n! \times \det\left[\frac{1}{(s_{j+1} - s_i)!}\right],$$

with $\frac{1}{(s_{j+1}-s_i)!} = 0$ if $j + 1 < i$.

Example. $\beta_3(\{2\}) = \#\{\sigma \in \mathfrak{S}_3 \mid \mathrm{shape}(\sigma) = \wedge\} = 3! \det \begin{bmatrix} 1/2 & 1 \\ 1/6 & 1 \end{bmatrix} = 2.$

The Peak Algebra

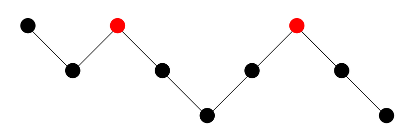
Given a permutation $\sigma \in \mathfrak{S}_n$, the **peaks** of σ is the set of positions

$$\mathrm{Peak}(\sigma) = \{i \in \{2, ..n-1\} \mid \sigma(i-1) < \sigma(i) > \sigma(i+1)\}.$$

It means that every subset S which doesn't contain $1, n$ and k if $k-1 \in S$ is a peak set and corresponds to at least one permutation.

From a descent set D, we can find the corresponding peak set by removing 1 and k if $k-1 \in D$. There is only one peak set corresponding to a descent set but in general there are more than one descent set corresponding to one peak set.

Example.



$\mathrm{Des}(\sigma) = \{1, 3, 4, 7, 8\}$
$\mathrm{Peak}(\sigma) = \{3, 7\}$

Definition 2. *For each peak set S of a permutation in $\mathfrak{S}_n$, we define $\mathcal{P}_n(S) = \sum_{\mathrm{Peak}(\sigma)=S} \sigma$.*

*The **peak algebra** $\mathcal{P}_n$ is the set spanned by $\mathcal{P}_n(S)$ for all peak sets S:*

$$\mathcal{P}_n = \mathrm{span}\{\mathcal{P}_n(S) \mid S \text{ is a peak set for } \mathfrak{S}_n\}.$$

Proposition 1. *$\mathcal{P}_n$ is a subalgebra of the descent algebra.*

$\mathcal{P}_n$ is a left ideal of $\mathcal{D}_n$

Let $\tau_{n,i}$ be the transposition switching i and $i+1$ in a permutation σ of $\mathfrak{S}_n$, with $\tau_{1,1}$ the identity in $\mathfrak{S}_1$.

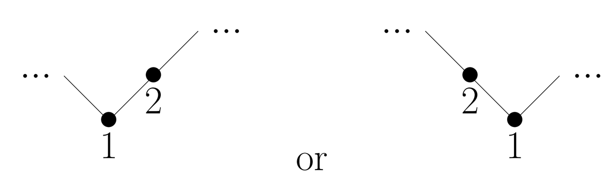
Example. $\tau_{5,1}(34512) = 34521$; $\tau_{5,3}(13254) = 14253$

We observe that for all $\sigma \in \mathfrak{S}_n$, swapping 1 and 2 doesn't change the peak set of σ:

$$\mathrm{Peak}(\sigma) = \mathrm{Peak}(\tau_{n,1}\sigma).$$

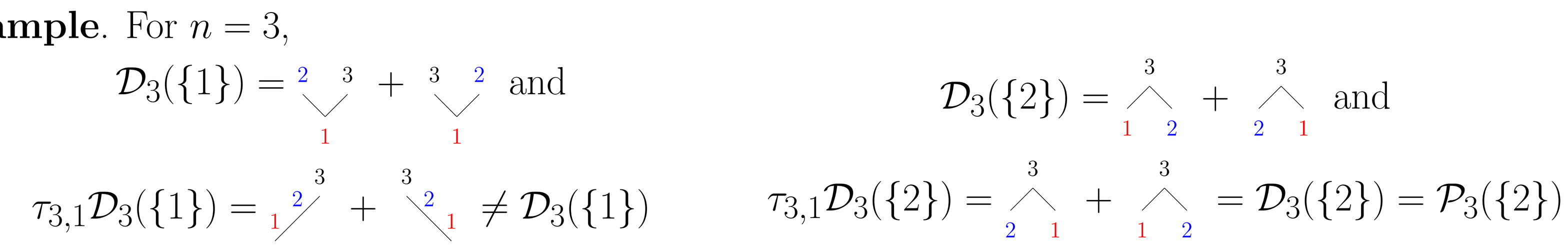
Indeed, recall that $\sigma^{-1}(i)$ is the position of i in σ. Then we have two different cases.

- If $|\sigma^{-1}(1) - \sigma^{-1}(2)| > 1$, the swap doesn't change the descent set and the peak set either.
- If $|\sigma^{-1}(1) - \sigma^{-1}(2)| = 1$, then 1 and 2 are preceded and followed by larger numbers and we have the two configurations on the right. Thus switching 1 and 2 does not change the peak set.



Theorem 1. *Let $P \in \mathcal{D}_n$. Then $P \in \mathcal{P}_n$ if and only if $\tau_{n,1}P = P$.*

Example. For $n = 3$,



This theorem also implies that $\mathcal{P}_n$ is a **left ideal** of $\mathcal{D}_n$.

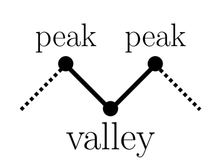
Example. For $n = 3$, we check that $\mathcal{P}_3$ is a left ideal of $\mathcal{D}_3$.
$\mathcal{D}_3 = \mathrm{span}\{\mathcal{D}_3(\varnothing) = 123, \mathcal{D}_3(\{1\}) = 213 + 312, \mathcal{D}_3(\{2\}) = 231 + 132, \mathcal{D}_3(\{1,2\}) = 321\}$
$\mathcal{P}_3 = \mathrm{span}\{\mathcal{P}_3(\varnothing) = 123 + 213 + 321 + 312, \mathcal{P}_3(\{2\}) = 231 + 132\}$

	$\mathcal{D}_3(\varnothing)$	$\mathcal{D}_3(\{1\})$	$\mathcal{D}_3(\{2\})$	$\mathcal{D}_3(\{1,2\})$
$\mathcal{P}_3(\varnothing)$	$\mathcal{P}_3(\varnothing)$	$\mathcal{P}_3(\varnothing) + 2\mathcal{P}_3(\{2\})$	$\mathcal{P}_3(\varnothing) + 2\mathcal{P}_3(\{2\})$	$\mathcal{P}_3(\varnothing)$
$\mathcal{P}_3(\{2\})$	$\mathcal{P}_3(\{2\})$	$\mathcal{P}_3(\varnothing)$	$\mathcal{P}_3(\varnothing)$	$\mathcal{P}_3(\{2\})$

Left Multiplication Table

Computing $\#\{\sigma \in \mathfrak{S}_n \mid \mathrm{Peak}(\sigma) = S\}$

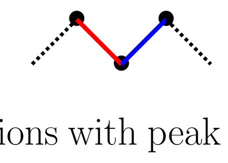
- For any pair of adjacent peaks, the sequence between them is the concatenation of a decreasing sequence and an increasing one. Thus, between any two peaks, one will always find a **valley**.



- For a given set of peaks and valleys (S, V), the number of permutations with peak and valley sets included in (S, V) is the number $\gamma_n(S)$ of ways to choose decreasing sequences going from a peak to a valley, and increasing sequences going from a valley to a peak. Assuming $|S| = k$,

$$\begin{aligned}\gamma_n(S) &= \sum_{1 \le v_0 < s_1 < v_1 < \ldots < v_{k-1} < s_k < v_k \le n} \gamma_n(S, V) \\ &= \sum_{1 \le v_0 < s_1 < v_1 < \ldots < v_{k-1} < s_k < v_k \le n} \binom{n}{v_0, s_1 - v_0, v_1 - s_1, s_2 - v_1, \ldots, s_k - v_{k-1}, v_k - s_k, n - v_k},\end{aligned}$$

with increasing and decreasing sequences:



- In order to compute the number of permutations with peak set S, we use inclusion-exclusion to exclude permutations such that $\mathrm{Peak}(\tau) \subsetneq S$:

$$\#\{\sigma \in \mathfrak{S}_n \mid \mathrm{Peak}(\sigma) = S\} = \sum_{T = \{s_{i_1}, \ldots, s_{i_j}\} \subseteq S} (-1)^{k-j} \gamma_n(T).$$

References

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[2] K.L. Nyman, *The Peak Algebra of the Symmetric Group*, Journal of Algebraic Combinatorics, Kluwer Academic Publishers, 2003.
[3] M. Schocker, *The peak algebra of the symmetric group revisited*, Advances in Mathematics, 2005.